MORE ON THE NORMALITY OF THE UNBOUNDED PRODUCT OF TWO NORMAL OPERATORS

MOHAMMED HICHEM MORTAD[1]* AND KHALDIA MADANI[2]

Abstract. Let A and B be two -non necessarily bounded- normal operators. We give new conditions making their product normal. We also generalize a result by Deutsch et al on normal products of matrices.

1. Introduction

First, we assume the reader is very familiar with notions, definitions and results on unbounded operators. All unbounded operators are assumed to be densely defined. Some general references are [1, 4, 8, 18, 19]. We just recall that an unbounded operator T is said to be normal if it is closed and $TT^* = T^*T$. We also note that between operators, the symbol "$\subset$" stands for extensions, i.e. $A \subset B$ means that $Ax = Bx$ for all $x \in D(A)$ and that $D(A) \subset D(B)$.

The question of when the product of two normal operators is normal is fundamental. For papers dealing with bounded normal products, see e.g. [7, 9, 17, 20, 21]. See also the recent paper [3] and the references therein for the bounded operators case. For the unbounded case, see [13, 15]. For closely related topics see [10, 11]. For those interested in sums of normal operators, see [12] and [16].

The following example illustrates that the passage from the bounded case to the unbounded one needs care.

Example 1. Let A be an unbounded normal operator having a trivial kernel, for example take $Af(x) = (1 + x^2)f(x)$ on $D(A) = \{f \in L^2(\mathbb{R}) : \ (1 + x^2)f \in L^2(\mathbb{R})\}$. Note that A is one-to-one but with properly dense range.

Now set $B = A^{-1}$. Observe that both A and B are normal on their respective domains (they are even self-adjoint and positive!). However BA, defined on $D(BA) = D(A)$, is not closed as $BA \subset I$. Thus it cannot be normal and yet B does commute with A.

For the reader's convenience, let us summarize, in a chronological order, all what has been obtained, to the authors best knowledge, as regards to the unbounded normal product of two operators:

Theorem 1 ([13]).

1. *Assume that B is a unitary operator. Let A be an unbounded normal operator. If B commutes with A (i.e. $BA \subset AB$), then BA is normal.*
2. *Assume that A is a unitary operator. Let B be an unbounded normal operator. If A commutes with B (i.e. $AB \subset BA$), then BA is normal.*

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*Corresponding author.

Dropping the unitarity hypothesis the following three results (also in [13]) were obtained:

Theorem 2. *Let B be a bounded normal operator. Let A be an unbounded normal operator. Assume that B commutes with A. If for some $r > 0$, $\|rBB^* - I\| < 1$, then BA is normal if it is closed.*

Theorem 3. *Let B a bounded normal operator and let A be an unbounded normal operator which commutes with B. Assume that for some $r > 0$, $\|rBB^* - I\| < 1$. Then AB is normal.*

Remark. Observe that the last two results generalize Theorem 1.

Proposition 1. *Let A be an unbounded normal operator and let B be a bounded normal operator commuting with A. If BB^* is strongly positive (in the sense given in* [5]*), then BA is normal.*

Very recently, in the context of generalizing Kaplansky's theorem (see [7]) one finds the following result. Of course, an assumption of unitarity on one of the operators is a strong one.

Theorem 4 ([15])**.** *If A is unitary and B is an unbounded normal operator, then*

$$BA \text{ is normal} \Longleftrightarrow AB \text{ is normal}.$$

In the present paper, we obtain new results by assuming that $AB = BA$ in lieu of $BA \subset AB$, under the conditions A and B both normal where only one of them is bounded. Then we show that an anti-commuting relation also gives a similar result. Then we show that in Theorem 2, the closedness of BA is not needed. Then we generalize a result by Deutsch et al which appeared in [2] to unbounded operators. Finally, we establish the normality of the product AB where both operators are unbounded.

To prove most of the results, we will make use of the following well-known results.

Lemma 1. *[*[8],[18]*] If S is (unbounded) symmetric and T is self-adjoint, then*

$$T \subset S \Longrightarrow T = S.$$

Lemma 2. *[*[8],[18]*] If T is closed, then T^*T and TT^* are both self-adjoint.*

Corollary 1. *If T is a closed operator such that $TT^* \subset T^*T$, then T is normal.*

Lemma 3 ([6] or [19])**.** *If A and B are densely defined and A is invertible with inverse A^{-1} in $B(H)$, then $(BA)^* = A^*B^*$.*

It is known that if B is bounded and A_1 and A_2 are unbounded and normal, then

$$BA_1 \subset A_2B \Longrightarrow BA_1^* \subset A_2^*B.$$

This is the well-known Fuglede-Putnam theorem. We can also derive the following version (also known but we include a proof for the reader's convenience) :

Theorem 5. *If B is bounded and A_1 and A_2 are unbounded and normal, then*

$$BA_1 = A_2B \Longrightarrow BA_1^* = A_2^*B.$$

Proof. By the Fuglede-Putnam theorem we have

$$BA_1 = A_2B \Longrightarrow BA_1 \subset A_2B \Longrightarrow BA_1^* \subset A_2^*B.$$

Hence $BA_1^* = A_2^*B$ for

$$D(A_2^*B) = D(A_2B) = D(BA_1) = D(A_1) = D(A_1^*) = D(BA_1^*).$$

□

A recently obtained generalization of the Fuglede-Putnam theorem is also valuable. It reads

Theorem 6 (Fuglede-Putnam-Mortad)**.** *Let A be a closed operator with domain $D(A)$. Let M and N be two unbounded normal operators with domains $D(N)$ and $D(M)$ respectively. If $D(N) \subset D(AN)$, then*

$$AN \subset MA \Longrightarrow AN^* \subset M^*A.$$

2. New Results

Here is the first result of the paper

Theorem 7. *Let A and B be two normal operators. Assume that B is bounded. If $BA = AB$, then BA (and so AB) is normal.*

Proof. Since $BA = AB$, by Theorem 5 we have $BA^* = A^*B$. Then we have

$$(BA)^*BA = A^*B^*BA = A^*B^*AB \underbrace{\subset}_{\text{classic Fuglede}} A^*AB^*B$$

and

$$BA(BA)^* = BAA^*B^* = ABA^*B^* = AA^*BB^* = A^*AB^*B.$$

Whence

$$(BA)^*BA \subset BA(BA)^*.$$

But BA is closed for it equals AB which is closed since A is closed and B is bounded. Therefore, $BA(BA)^*$ and $(BA)^*BA$ are both self-adjoint (by Lemma 2) and hence BA is normal (by Corollary 1), completing the proof. □

Remark. The assumption $AB \subset BA$ cannot merely be dropped. By Example 1,

$$D(AB) = L^2(\mathbb{R}) \not\subset D(BA) = D(A) = \{f \in L^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R})\}.$$

We also obtain an "anti-commuting version" of Theorem 7.

Theorem 8. *Let A and B be two normal operators. Assume that B is bounded. If $BA = -AB$, then BA (and so AB) is normal.*

Proof. The same idea of proof as that of the previous result applies. We have $BA^* = -A^*B$ thanks to Theorem 5 because $-A$ is also normal. Then

$$(BA)^*BA = A^*B^*BA = -A^*B^*AB \underbrace{\subset}_{\text{Fuglede}} A^*AB^*B$$

and

$$BA(BA)^* = BAA^*B^* = -ABA^*B^* = AA^*BB^* = A^*AB^*B.$$

The rest is obvious. □

Now, we improve Theorem 2 by removing the assumption that BA be closed.

Theorem 9. *Let B be a bounded normal operator. Let A be an unbounded normal operator. Assume that B commutes with A. If for some $r > 0$, $\|rBB^* - I\| < 1$, then BA is normal.*

Proof. The proof is the same as the one in [13]. What we are concerned with here is to show that the closedness of BA is tacitly assumed.

So let us show that BA is closed. Let $x_n \to x$ and $BAx_n \to y$. The condition $\|rBB^* - I\| < 1$, plus the normality of B, guarantees that $BB^* = B^*B$ is invertible. Hence, by the continuity of B^*, $B^*BAx_n \to B^*y$. Therefore,

$$Ax_n \longrightarrow (B^*B)^{-1}B^*y.$$

But A is closed, hence $x \in D(A)$ and $Ax = (B^*B)^{-1}B^*y$. This implies that

$$B^*BAx = B^*y \text{ and hence } BB^*BAx = BB^*y$$

which, thanks to the invertibility of BB^*, clearly yields $BAx = y$, proving the closedness of BA. □

Next, we give an unbounded oeprator version of a result by Deutsch et al in [2] (cf. [20] and [21]) on normal products of matrices. We have

Theorem 10. *Let A be a bounded and invertible operator. Let B be unbounded and closed. Assume further that $D(B) \subset D(BAB)$. Then BA and AB are normal iff $BAA^* = A^*AB$ and $B^*BA \subset ABB^*$.*

Proof. First, we note that we should not worry about the closedness of both BA and AB for the boundedness and the invertibility of A (and the closedness of B!) implies that BA and AB are closed respectively.

(1) Assume that $BAA^* = A^*AB$ and $B^*BA \subset ABB^*$ and let us show that BA and AB are normal. Since A is invertible, Lemma 3 implies that $(BA)^* = A^*B^*$, and also

$$B^*BA \subset ABB^* \Longrightarrow BB^*A^* \subset A^*B^*B,$$

where we also used Lemma 2 for B. Hence

$$(BA)^*BA = A^*B^*BA \supset BB^*A^*A.$$

So by using again the invertibility of A (and hence that of A^*A) and Lemma 2 we obtain

$$(BB^*A^*A)^* = A^*ABB^* \subset ((BA)^*BA)^* = (BA)^*BA.$$

On the other hand, we see that

$$BA(BA)^* = BAA^*B^* = A^*ABB^*$$

which implies that

$$BA(BA)^* \subset (BA)^*BA.$$

Corollary 1 then makes the "inclusion" an exact equality, i.e. establishing the normality of BA.

Let us turn now to the product AB. This is more straightforward. We have

$$(AB)^*AB = B^*A^*AB = B^*BAA^*$$

and

$$AB(AB)^* = ABB^*A^* \supset B^*BAA^*.$$

Arguing similarly as before gives the normality of AB. This finishes the first part of the proof.

(2) Assume that BA and AB are both normal. Then

$$A(BA) = (AB)A \Longrightarrow A(BA)^* = (AB)^*A \Longrightarrow AA^*B^* = B^*A^*A$$

by Theorem 5 and the invertibility of A.

We also have

$$B(AB) = (BA)B \Longrightarrow B(AB) \subset (BA)B \Longrightarrow B(AB)^* \subset (BA)^*B$$

by Theorem 6 (since $D(B) \subset D(BAB)$) and the boundedness of A. Hence

$$BB^*A^* \subset A^*B^*B \text{ or } B^*BA \subset ABB^*$$

and the proof is then complete.

□

Consider next the following example:

Example 2. Let A and B be the two operators defined by

$$Af(x) = e^{ix}f(x) \text{ and } Bf(x) = e^{x^2-ix}f(x)$$

on their respective domains

$$D(A) = L^2(\mathbb{R}) \text{ and } D(B) = \{f \in L^2(\mathbb{R}) : \ e^{x^2}f(x) \in L^2(\mathbb{R})\}.$$

Then A is unitary (so $BAA^* = A^*AB$ is verified) and B is normal. Moreover, we can easily check that:

$$D(B^*BA) = \{f \in L^2(\mathbb{R}) : \ e^{2x^2}f(x) \in L^2(\mathbb{R})\}$$

and

$$D(ABB^*) = D(BB^*) = \{f \in L^2(\mathbb{R}) : \ e^{2x^2}f(x) \in L^2(\mathbb{R})\}$$

too. Since

$$B^*BAf(x) = ABB^*f(x), \ \forall f \in D(B^*BA) = D(ABB^*),$$

we have $B^*BA = ABB^*$. We also see that both AB and BA are normal on their equal domains

$$D(AB) = D(BA) = \{f \in L^2(\mathbb{R}) : \ e^{x^2}f(x) \in L^2(\mathbb{R})\}$$

since they are the multiplication operator by the function e^{x^2}. Nonetheless we have

$$D(BAB) = \{f \in L^2(\mathbb{R}) : \ e^{2x^2}f(x) \in L^2(\mathbb{R})\}$$

and so $D(B) \not\subset D(BAB)$ as, for instance, $e^{-\frac{3}{2}x^2} \in D(B)$ but $e^{-\frac{3}{2}x^2} \notin D(BAB)$.

This example suggests that replacing "bounded and invertible" by "unitary" might allow us to drop the condition $D(B) \subset D(BAB)$ there. This is in fact the case and we have

Theorem 11. *Let A be a unitary operator. Let B be unbounded and closed. Then BA and AB are normal iff $B^*BA \subset ABB^*$.*

Proof. The proof of sufficiency is as before. Note that with A assumed unitary, the first condition of Theorem 10 is automatically satisfied.

Let us suppose that BA and AB are both normal and let us check that $B^*BA \subset ABB^*$. In fact, since AB is normal, we have

$$(AB)^*AB = B^*A^*AB = B^*B = AB(AB)^* = ABB^*A^*.$$

Hence $BB^*A^* = A^*B^*B$. Accordingly by taking adjoints,

$$ABB^* = B^*BA,$$

establishing the result. □

We now turn to the case of two unbounded normal operators. We have

Theorem 12. *Let A be an unbounded invertible normal operator. Let B be an unbounded normal operator. If $BA = AB$, $A^*B \subset BA^*$ and $B^*A \subset AB^*$, then BA is normal.*

Proof. Since A is invertible, by Lemma 3, $(BA)^* = A^*B^*$. Then

$$(BA)^*BA = A^*B^*BA = A^*B^*AB \subset A^*AB^*B$$

and

$$BA(BA)^* = BAA^*B^* = BA^*AB^* \supset A^*BAB^* = A^*ABB^*.$$

Therefore,

$$(BA)^*BA \subset BA(BA)^*.$$

Since $BA = AB$, A is invertible and closed, and B is closed, BA is closed and Lemma 1 does the remaining job, i.e. gives us:

$$(BA)^*BA = BA(BA)^*,$$

completing the proof.

□

The same method of proof yields

Theorem 13. *Let A be an unbounded invertible normal operator. Let B be an unbounded normal operator. If $BA \subset AB$, $A^*B \subset BA^*$ and $B^*A \subset AB^*$, then BA is normal whenever it is closed.*

Finally, adopting the same idea of the proof of Theorem 12 and using Theorem 6, we can impose some conditions on domains to derive a domains-dependent version of Theorem 1.

Corollary 2. *Let A and B be two unbounded invertible normal operators with domains $D(A)$ and $D(B)$ respectively. If $BA = AB$ and $D(A), D(B) \subset D(BA)$, then BA (and AB) is normal.*

Proof. Note first that the closedness of BA is clear. Now we have

$$BA \subset AB \Longrightarrow BA^* \subset A^*B \Longrightarrow B^*A \subset AB^*$$

by $D(A) \subset D(BA)$, Theorem 6 and the invertibility of A^*. Similarly, we have

$$AB \subset BA \Longrightarrow AB^* \subset B^*A \Longrightarrow A^*B \subset BA^*$$

by $D(B) \subset D(AB)$, Theorem 6 and the invertibility of B^*. So we came back to the setting of Theorem 12. □

3. Conclusion

New results for the normality of the unbounded product of two normal operator, have been obtained. A result by Deutsch et al for normal matrix products has been generalized to general and unbounded products.

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[1]Department of Mathematics, University of Oran, B.P. 1524, El Menouar, Oran 31000. Algeria.

Mailing address (for the first author):
Dr Mohammed Hichem Mortad
BP 7085 Es-Seddikia
Oran
31013
Algeria

E-mail address: mhmortad@gmail.com, mortad@univ-oran.dz.

[2]Laboratoire LTE, ENSET, El Menouar B.P. 1523, Es-Senia, Oran 31000. Algeria.

E-mail address: madanikhaldia@gmail.com